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A combinatorial understanding of lattice path asymptotics

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We provide a combinatorial derivation of the exponential growth constant for counting sequences of lattice path models restricted to the quarter plane. The values arise as bounds from analysis of related half planes models. We give explicit formulas, and the bounds are provably tight. The strategy is easily generalizable to cones in higher dimensions, and has implications for random generation.

1. Introduction

Lattice path models have enjoyed a sustained popularity in mathematics over the past century, owing in part to their simplicity and ease of analysis, but also their wide applicability both in mathematics, physics, and chemistry. The basic enumerative question is

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to determine the number of walks of a given length in a given model. The past fifteen years have seen many interesting developments in the asymptotic and exact enumeration of lattice models, with new techniques coming from computer algebra, complex analysis and algebra. A first approximation to this value is the *exponential growth constant*, also called the *connective constant*, which itself carries combinatorial and probabilistic information. For example, it is directly related to the limiting free energy in statistical mechanical models.

Here, a lattice path model is defined by the steps that are allowed, and the region to which the walks are restricted (generally cones and strips). Particular focus has been on small step models (where the steps are a subset of $\{0, \pm 1\}^2$) restricted to $\mathbb{Z}^2_{\geq 0}$, and general approaches versus resolution of individual cases. For example, several distinct strategies for asymptotic enumeration have recently emerged. Fayolle and Raschel [10] have determined expressions for the growth constant for small step models using boundary value problem techniques. Recast as diagonals, techniques of analytic combinatorics of several variables apply to some of the models with D-finite generating functions [15, 16]. Also, the important sub-class of excursions is well explored via the probability work of Denisov and Wachtel [7, Section 1.5]. Bostan, Raschel and Salvy [5] made their results explicit in the enumeration context. Most of these asymptotic results are obtained with machinery which does not sustain a clear underlying combinatorial picture.

Many of these results exclude *singular* models: A two dimensional model is singular if the support of the step set is contained in a half plane. Many singular models are either trivial or reduce to a problem in a lower dimension. Singular models are considered in [17,14].

This paper provides a formula for an upper bound on the growth constant of the counting sequence for lattice models restricted to a convex cone, with intuitive combinatorial interpretations of intermediary computations. Our formula is most explicit in the case of nonsingular 2-dimensional walks restricted to the first quadrant, but is valid *for all models*. The core of our strategy is based on the following basic observation:

In any lattice path model, the set of walks restricted to the first quadrant is a subset of the walks restricted to some half plane which contains that quadrant. Consequently, for any fixed length, the number of walks in that half plane is an upper bound for the number of walks in the quarter plane.

Bounds on walks in half planes are readily computable, for example using the results of Banderier and Flajolet [1]. Remarkably, we are able to give *tight* bounds on the growth constant by considering all of the half planes that contain the quarter plane. Furthermore, our bounds are insightfully tight in that they give a single simple combinatorial interpretation of the multiple cases treated by Fayolle and Raschel [10]. Our one idea *unifies* their cases, which depend on various parameters of the model. Our approach also applies to singular models. We use only the elementary calculus observation that a minimum of a real valued differentiable function f with domain D must occur either at the boundary of D or at a critical point $\tau \in D$ satisfying $f'(\tau) = 0$. This strategy remains combinatorial and readily adaptable to models with larger steps, weighted steps, and to models in higher dimensions. Finally, there are implications for random generation, as we discuss in the conclusion.

In an earlier version of this article we conjectured that our bounds were tight. This led to a proof by Garbit and Raschel [11] that the bounds we find are tight under some conditions. These are quite general conditions, and include at least all of the nonsingular walks, for example. Simultaneously, and independently, similar results were proved by Duraj [9].

1.1. Conventions and notation

Let R be a convex cone in $\mathbb{R}^d$, and let $\mathcal{S}$ be a finite multiset of vectors in $\mathbb{R}^d$. Here, a walk is a sequence of steps, each step taken from $\mathcal{S}$, starting at the origin. The set of walks of length n restricted to R is defined as the set

$$\mathrm{R}(\mathcal{S})_n = \{(s_1, s_2, \ldots, s_n) : s_i \in \mathcal{S}; \sum_{i=1}^{j} s_i \in \mathrm{R} \text{ for } 1 \leq j \leq n\}.$$

The full set of walks, denoted $\mathrm{R}(\mathcal{S})$, is the union of these sets over all natural numbers n. By convention, we permit a single empty walk of length 0.

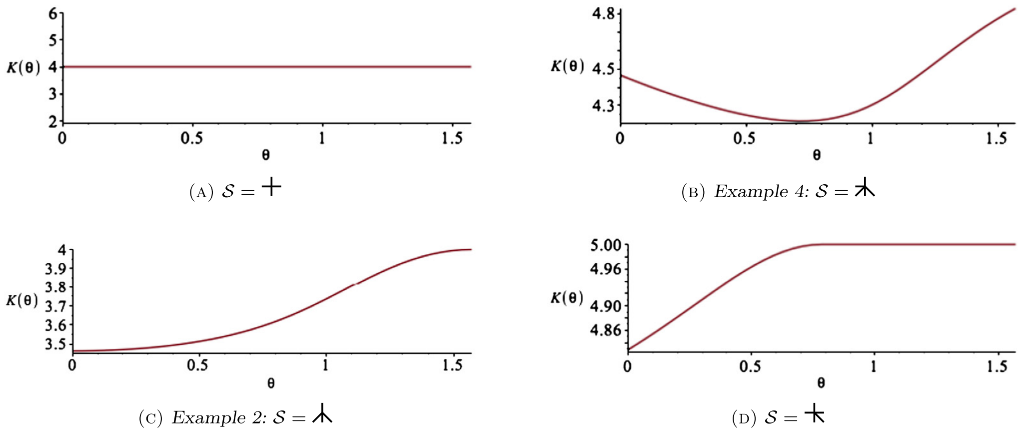
The cones we consider include half planes through the origin, and the first quadrant $\mathbb{Z}^2_{\geq 0}$. The central quantity we investigate is the number of these walks, $|\mathrm{R}(\mathcal{S})_n|$. We write $\mathrm{H} = \mathbb{R} \times \mathbb{R}_{\geq 0}$ for the upper half plane and $\mathrm{Q} = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ for the first quadrant, and abbreviate $h_n = |\mathrm{H}(\mathcal{S})_n|$ and $q_n = |\mathrm{Q}(\mathcal{S})_n|$ when $\mathcal{S}$ is clear.

We say a model $\mathrm{R}(\mathcal{S})$ is *nontrivial* if it contains at least one walk of positive length, and if for every boundary of R, there exists an unrestricted walk on $\mathcal{S}$ which crosses that boundary at some point other than an intersection of boundaries (i.e. in two dimensions, not at the origin). The *excursions* are the sub-class consisting of walks which start and end at the origin. A step set is said to be *singular* if it is contained within a single half plane. A step set is said to be made of *small steps* if $\mathcal{S} \subseteq \{0, \pm 1\}^2 \setminus \{(0,0)\}$ and in this case we use the compass abbreviations $NW \equiv (-1, 1), N \equiv (0, 1), NE \equiv (1, 1)$, etc. We also consider larger regions, and more general step sets in the examples.

The (exponential) *growth constant* of a sequence (a_n) is defined as the limit $\lim_{n\to\infty} a_n^{1/n}$, when it exists. This limit exists as a consequence of Fekete's subadditivity lemma, see Hille [12]. It suffices to observe that the counting sequence is supermultiplicative, that is,

$$a_n \cdot a_m \leq a_{n+m}.$$

Counting sequences for lattice models in cones satisfy this relation since a walk of length $m + n$ can not necessarily be decomposed as a walk of length m followed by a walk of length n that both remain in the cone. Indeed, equality $a_n \cdot a_m = a_{n+m}$ does not hold in general.



(A) $\mathcal{S} =$ (B) *Example 4:* $\mathcal{S} =$ (C) *Example 2:* $\mathcal{S} =$ (D) $\mathcal{S} =$

Fig. 1. Graphs of $K_{\mathcal{S}}(\theta)$ for various $\mathcal{S}$.

We denote the growth constant of quarterplane walks by

$$K_{\mathcal{S}} = \lim_{n\to\infty} q_n^{1/n}.$$

The present work determines bounds for this growth constant.

Our strategy uses the simple relation that if $\mathrm{Q} \subset \mathrm{R}$, for some cone R, then $\mathrm{Q}(\mathcal{S})_n \subset \mathrm{R}(\mathcal{S})_n$ and hence $q_n \leq |\mathrm{R}(\mathcal{S})_n|$. This is true for all n, hence it is also true that $\lim_n q_n^{1/n} \leq \lim_n |\mathrm{R}(\mathcal{S})_n|^{1/n}$. It turns out, by considering well chosen regions, we are able to perfectly bound the growth constant $K_{\mathcal{S}}$. Our preferred bounding regions are the half planes

$$\mathrm{H}_\theta = \{(x, y) : x \sin\theta + y \cos\theta \geq 0\}$$

where $\theta \in [0, \pi/2]$. We denote by $K_{\mathcal{S}}(\theta)$ the growth constant of the sequence of the number of walks of length n in this region:

$$K_{\mathcal{S}}(\theta) = \lim_{n\to\infty} |\mathrm{H}_\theta(\mathcal{S})_n|^{1/n}.$$

Using the relation

$$K_{\mathcal{S}} \leq K_{\mathcal{S}}(\theta) \quad \text{for all } 0 \leq \theta \leq \pi/2,$$

we can deduce bounds on $K_{\mathcal{S}}$ by finding explicit expressions for $K_{\mathcal{S}}(\theta)$. Fig. 1 gives several examples for several different stepsets.

1.2. The main result and the plan of the paper

Our main result, Theorem 8, is an explicitly computable bound on $K_{\mathcal{S}}$, and its combinatorial interpretation. It is determined by minimizing $K_{\mathcal{S}}(\theta)$ as a function of θ. In several cases it is easily seen to be tight, by comparing to the well-understood subclass of excursions.

We start at Lemma 7, where we adapt the formulas of Banderier and Flajolet to give a formula for $K_{\mathcal{S}}(\theta)$, for given $\mathcal{S}$ and θ. We then show that $K_{\mathcal{S}}(\theta)$ defines a continuous function in θ. Since each H_θ contains Q, $K_{\mathcal{S}}(\theta)$ is an upper bound on $K_{\mathcal{S}}$ for any θ satisfying $0 \leq \theta \leq \pi/2$. Finally, we determine the location of the minimum upper bound in Theorem 8 by basic calculus techniques, since $K_{\mathcal{S}}(\theta)$ is an explicit function of θ.

In Section 4, we show that these results give precisely the values found by Fayolle and Raschel for the nonsingular models, demonstrating the bounds are tight. It is Theorem 12 which vindicates the description of this work as a combinatorial interpretation of the formulas provided by Fayolle and Raschel.

Our strategy applies to more general classes of models, for example, multiple steps in the same direction, longer steps, and higher dimensional models. The quantities we recover in these cases are, transparently, upper bounds and they can be compared against experimental data as a check for tightness. This led us to conjecture that our approach gives tight upper bounds more generally and hence actually finds the growth constants, which has subsequently been proven under some general hypotheses [11, Corollary 10] through probabilistic arguments.

2. Walks in a half plane

Models restricted to a half plane are well understood, and we recall here some basic results. The set $\mathrm{H}(\mathcal{S})$ of walks restricted to the upper half plane with steps from the finite multiset $\mathcal{S}$ is in bijection with unidimensional walks with steps from the multiset $\mathcal{A} = \{j : (i,j) \in \mathcal{S}\}$ because horizontal movement does not lead to any interaction with the boundary of H. We thus consider half plane models as unidimensional models defined by sets of real numbers. We retain the same notation, and meaning:

$$\mathrm{H}(\mathcal{A})_n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathcal{A}; \sum_{i=1}^{j} a_i \geq 0, 1 \leq j \leq n\}$$

denotes the set of walks of length n starting at 0. The multiset $\mathcal{A}$ is said to be *nontrivial*, if it contains at least one positive and one negative value. There are two ways for a multiset $\mathcal{A}$ to be trivial in a half plane: either $\mathcal{A}$ contains only non-negative elements and we call it *unrestricted*; or $\mathcal{A}$ contains only non-positive elements. Unless otherwise stated, we assume that the models are nontrivial. It is worth noting that Theorem 1 still holds in the unrestricted case.

The key ingredients for the enumeration formulas are as follows. The *drift* of $\mathcal{A}$ is the sum $\delta(\mathcal{A}) = \sum_{a \in \mathcal{A}} a$ and the *inventory* of $\mathcal{A}$ is $A(u) = \sum_{a \in \mathcal{A}} u^a$; notice that these are related by $\delta(\mathcal{A}) = A'(1)$.

Theorem 1 *(Modified from Theorem 4 of Banderier and Flajolet [1]). Let $\mathcal{A}$ be a multiset of integers which defines a nontrivial unidimensional walk model. Let $A(u) = \sum_{a\in\mathcal{A}} u^a$. The number h_n of walks of length n in $\mathrm{H}(\mathcal{A})$ depends on the sign of the drift $\delta(\mathcal{A}) = A'(1)$ as follows:*

$$h_n \sim \begin{cases} \nu_0\, A(1)^n & \text{if } \delta(\mathcal{A}) > 0 \\ \nu_1\, A(1)^n\, n^{-1/2} & \text{if } \delta(\mathcal{A}) = 0 \\ \nu_2\, A(\tau)^n\, n^{-3/2} & \text{if } \delta(\mathcal{A}) < 0 \end{cases}.$$

Here τ is the unique positive critical point of $A(u)$ and ν_0, ν_1 and ν_2 are explicit, real constants.

Proof. This follows directly from [1]. Remark the case of unrestricted walks with steps from $\mathcal{A}$, $h_n = |\mathcal{A}|^n = A(1)^n$ so setting $\nu_2 = 1$ gives the result. □

The study of the growth constant originates in the probability literature [8]. On the enumerative side, Banderier and Flajolet prove these formulas by applying transfer theorems to explicit generating functions, which they first derive. The strategy requires integer steps however: The models with real-valued steps are not necessarily representable by context-free grammars, and the generating functions are not necessarily algebraic. It is rather straightforward to deduce the general case from the integer case by a limit argument which we present next.

Theorem 2. *Let $\mathcal{A}$ be a multiset of real numbers which defines a nontrivial unidimensional walk model. Let $A(u) = \sum_{a\in\mathcal{A}} u^a$. The number $K_{\mathcal{A}} = \lim_{n\to\infty} h_n^{1/n}$, where h_n is the number of walks of length n, depends on the sign of the drift $\delta(\mathcal{A}) = A'(1)$ as follows:*

$$K_{\mathcal{A}} = \begin{cases} |\mathcal{A}| & \text{if } \delta(\mathcal{A}) \geq 0 \\ A(\tau) & \text{otherwise.} \end{cases} \tag{2.1}$$

Here τ is the unique positive critical point of $A(u)$.

First a few comments about τ. Since $\delta(\mathcal{A}) = A'(1)$, by uniqueness of τ, if $\delta(\mathcal{A}) = 0$ then $\tau = 1$. Similarly, if $\delta(\mathcal{A}) < 0$, then $\tau > 1$ by the convexity of $A(u)$ (derived below). When $\delta(\mathcal{A}) > 0$, then $\tau < 1$, but $K_{\mathcal{A}} = |\mathcal{A}| = A(1)$. We can combine these three cases into a single equation since τ is a unique local minimum. We rewrite Equation (2.1) as

$$K_{\mathcal{A}} = \min_{u\geq 1} A(u). \tag{2.2}$$

This could be viewed as a one dimensional analogue of our main formula.

It should be emphasized here that these formulas all still hold in the unrestricted case. In that case, the drift is nonnegative, and $h_n = |\mathcal{A}|^n$. Thus, the theorem is also true under weaker hypotheses. Theorem 1 establishes Formula 2.1 for $\mathcal{A} \subset \mathbb{Z}$. The proof for other real nontrivial models $\mathcal{A} \subset \mathbb{R}$ is established from the integer base case in three steps:

1. We show that if Equation (2.1) holds for the multiset $\mathcal{A}$, then it is also true for the multiset $r\mathcal{A} = \{ra : a \in \mathcal{A}\}$ when $r > 0$ in Lemma 3;
2. We then deduce that the formula holds for multisets of rationals in Remark 4;
3. Finally, we prove that the formula holds for multisets of reals by proving that a limiting construction of rational models gives the result. This is done in Section 2.3.

We remark that the growth constant of the sequence counting the number of excursions of length n in H (in the integer case) can be shown to be $A(\tau)$ using a strategy similar to the proof of Theorem 1 [1, Theorem 3]. The excursion formulas for integers do not automatically become formulas for the general real case, since if the reals are not rational multiples of each other, there are no excursions. To determine a formula, the reals much first be partitioned into subsets where they satisfy some integer relations.

2.1. Some facts about the inventory $A(u)$

The first two steps of the proof of Theorem 2 follow from basic behavior of $A(u)$.

Lemma 3 *(Scaling lemma). Let $\mathcal{A}$ be a finite multiset of real numbers which is either unrestricted or nontrivial and let $A(u) = \sum_{a\in\mathcal{A}} u^a$. Suppose further that Equation (2.1) holds for $K_{\mathcal{A}}$. For any $r > 0$, define $\mathcal{B} = r\mathcal{A} = \{ra : a \in \mathcal{A}\}$. Then the growth constant $K_{\mathcal{B}}$ of the sequence $b_n = |\mathrm{H}(\mathcal{B})_n|$ satisfies*

$$K_{\mathcal{B}} = \begin{cases} |\mathcal{B}| & \text{if } \delta(\mathcal{B}) \geq 0 \\ B(\tau_{\mathcal{B}}) & \text{otherwise} \end{cases}. \tag{2.3}$$

Here $B(u) = \sum_{b\in\mathcal{B}} u^b$ and $\tau_{\mathcal{B}}$ is the unique positive critical point of $B(u)$.

Proof. The lattice model $\mathrm{H}(\mathcal{B})$ is combinatorially isomorphic to $\mathrm{H}(\mathcal{A})$, so their growth constants are the same. The formula follows because their drifts have the same sign, and since $B(u) = A(u^r)$ and $\tau_{\mathcal{B}} = \tau_{\mathcal{A}}^{1/r}$, thus $B(\tau_{\mathcal{B}}) = A(\tau_{\mathcal{A}})$. □

Remark 4. For any finite multiset of rational numbers $\mathcal{B}$, there is an $r > 0$, for example, the least common multiple of the denominators in $\mathcal{B}$, so that $\mathcal{A} = r\mathcal{B}$ is a multiset of integers. By Theorem 1, $K_{\mathcal{A}}$ satisfies Equation (2.1), and hence Equation (2.3). Consequently, by Lemma 3, since $\mathcal{B} = \frac{1}{r}\mathcal{A}$, it is also true that $K_{\mathcal{B}}$ satisfies Equation (2.3).

Lemma 5 *($A(u)$ is strictly convex at its minimum). Given a finite multiset $\mathcal{A}$ of real numbers which defines a nontrivial unidimensional model, the real valued function $A(u) = \sum_{a\in\mathcal{A}} u^a$ has a unique positive critical point τ. The function is minimized* *at this point, and is strictly convex on a neighborhood of* τ*. Furthermore, if* $\delta(\mathcal{A}) = 0$*, then the unique critical point occurs at* $u = 1$*, and if* $\delta(\mathcal{A}) < 0$ *then it occurs at* $u > 1$*.*

Proof. If there are no elements in the range $(0, 1)$ then the convexity result holds term by term. Otherwise it is possible to scale so that all elements are outside of that range.

Finally, if $\delta(\mathcal{A}) = 0$, then $A'_{\mathcal{A}}(1) = 0$. Thus, by the uniqueness of the positive critical point, $\tau_{\mathcal{A}} = 1$. By similar analysis the negative drift case holds. □

2.2. The continuity of $A(\tau_{\mathcal{A}})$ *as a function of* $\mathcal{A}$

To prove Theorem 2, we consider the function $A(u)$, evaluated at its critical point. In particular we view this as a function of the step lengths. In the following lemma, ℓ represents the number of elements in the step set.

Lemma 6. *Let* $F : \mathbb{R}^\ell \times \mathbb{R}_{>0} \to \mathbb{R}$ *be the function defined by*

$$F((x_1, x_2, \ldots, x_\ell), u) = F(\mathbf{x}, u) = \sum_{j=1}^{\ell} u^{x_j}.$$

Furthermore, let $\mathbf{a} \in \mathbb{R}^\ell$ *have at least one positive component and one negative component, and denote by* $\tau(\mathbf{a})$ *the unique positive critical point of the map* $u \mapsto F(\mathbf{a}, u)$*. There is a neighborhood* $\mathcal{U}$ *of* $\mathbf{a}$ *such that the function*

$$\kappa(\mathbf{x}) = F(\mathbf{x}, \tau(\mathbf{x})),$$

is continuous in $\mathbf{x}$ *on* $\mathcal{U}$*.*

Proof. The results is a consequence of the implicit function theorem applied to $f(\mathbf{x}, u) = \frac{\partial F(\mathbf{x},u)}{\partial u}$ around the point $\mathbf{a}$. □

2.3. Proof of Theorem 2 in the case of real multisets

Proof of Theorem 2. Let $\mathcal{A} \subset \mathbb{R}$ be a finite multiset of real numbers which is either unrestricted or nontrivial. To prove that $K_{\mathcal{A}}$ satisfies Equation (2.1) we build two sequences of rational step sets which converge to $\mathcal{A}$. We then squeeze the growth constant $K_{\mathcal{A}}$ of $h_n = |\mathrm{H}(\mathcal{A})_n|$ into the desired form.

For each $a \in \mathcal{A}$, let $\{a_i^+\}$ and $\{a_i^-\}$ be rational sequences satisfying

$$0 \leq a_i^+ - a \leq \frac{1}{2^i} \qquad \text{and} \qquad 0 \leq a - a_i^- \leq \frac{1}{2^i}.$$

We define two multisets

$$\mathcal{A}_i^+ = \{a_i^+ : a \in \mathcal{A}\} \qquad \text{and} \qquad \mathcal{A}_i^- = \{a_i^- : a \in \mathcal{A}\}.$$

The drift is additive, thus for each i,

$$\delta(\mathcal{A}_i^-) = \sum_{a\in\mathcal{A}} a_i^- \leq \sum_{a\in\mathcal{A}} a \leq \sum_{a\in\mathcal{A}} a_i^+ = \delta(\mathcal{A}_i^+).$$

Note that Remark 4 applies to both $\mathcal{A}_i^-$, and $\mathcal{A}_i^+$ because both are multisets of rational numbers and hence (2.1) is valid for the growth constants, $K_{\mathcal{A}_i^-}$ and $K_{\mathcal{A}_i^+}$ respectively.

Given a one dimensional walk which remains in the upper half plane, a walk produced by lengthening every positive step slightly, and shortening every negative step, will also remain in the upper half plane. Thus, by the construction of $\mathcal{A}_i^+$ and $\mathcal{A}_i^-$ we get a natural injection

$$\mathrm{H}(\mathcal{A}_i^-) \hookrightarrow \mathrm{H}(\mathcal{A}) \hookrightarrow \mathrm{H}(\mathcal{A}_i^+)$$

and hence

$$K_{\mathcal{A}_i^-} \leq K_{\mathcal{A}} \leq K_{\mathcal{A}_i^+}. \tag{2.4}$$

We claim

$$\lim_{i\to\infty} K_{\mathcal{A}_i^-} = K_{\mathcal{A}} = \lim_{i\to\infty} K_{\mathcal{A}_i^+},$$

and $K_{\mathcal{A}}$ is given by the formula of Theorem 1. To prove this claim observe the following. In all cases $\mathcal{A}_i^\pm \to \mathcal{A}$ and hence $\delta(\mathcal{A}_i^\pm) \to \delta(\mathcal{A})$.

In the case of $\mathcal{A}$ of positive drift, the result is clear, since $K_{\mathcal{A}}$ is squeezed between two (eventually) constant sequences with the same limit, namely, $|\mathcal{A}|$.

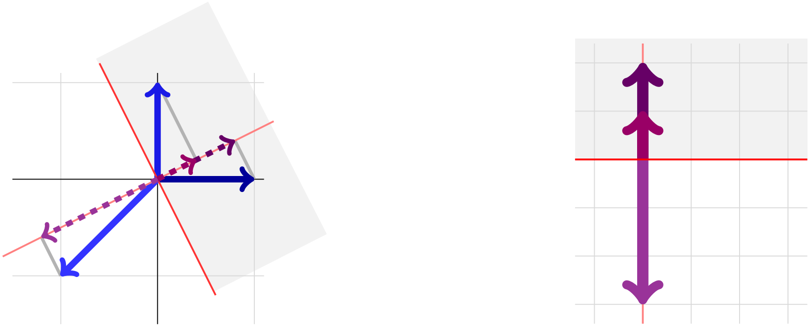
In the case where $\mathcal{A}$ is of negative drift, we appeal to Lemma 6. For sufficiently large i the values of $K_{\mathcal{A}_i^-}$ and $K_{\mathcal{A}_i^+}$ are merely evaluations of κ. Consequently, both limits $\lim_{i\to\infty} K_{\mathcal{A}_i^\pm}$ exist, and are the same value, by continuity. By Equation 2.4 and the squeeze theorem, we have that $K_{\mathcal{A}}$ is also an evaluation of κ.

Finally, in the case $\delta(\mathcal{A}) = 0$ we verify that the lower limit and the upper limit agree. The upper limit is $|\mathcal{A}|$ since the drift is nonnegative. The lower limit is given by a set of walks with negative step set, whose exponential growth is given by κ evaluated at the critical point. Since the drift of these models is tending to 0, the critical points are tending to 1 (Lemma 5). The inventory evaluated at 1 is the number of terms, precisely $|\mathcal{A}|$.

In summary,

$$\lim_{i\to\infty} K_{\mathcal{A}_i^\pm} = \begin{cases} |\mathcal{A}| & \text{if } \delta(\mathcal{A}) \geq 0 \\ A(\tau_{\mathcal{A}}) & \text{otherwise} \end{cases}$$

and the result follows by the squeeze theorem. □



(A) *The step set $\mathcal{S} = \{N, E, SW\}$ (in blue), and its projection onto the line $y = x/2$ (in purple, dashed)*

(B) *Step set for the unidimensional model $H(\{1, 2, -3\})$*

Fig. 2. Three representations for models of walks with steps from $\mathcal{S} = \{N, E, SW\}$ restricted to the region $\{y \geq -2x\}$, defined by $\theta = \arctan(2)$: (A) $H_\theta(\mathcal{S})$, and its unidimensional projection and (B) $H(\{1, 2, -3\})$, a scaling of $\mathcal{A}(\theta)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.4. Other half planes

Next, we extend to other half planes, each defined by an angle: $\mathrm{H}_\theta = \{(x, y) : x \sin\theta + y \cos\theta \geq 0\}$. Note that for $\theta \in [0, \pi/2)$ this region is equal to $\{(x, y) : y \geq -mx\}$ where $m = \tan\theta$. The upper half plane is given by H_0 and the right half plane is $\mathrm{H}_{\pi/2}$. In this latter case, we use the extended reals, and write $m = \infty$. The enumeration of lattice paths in H_θ emulates the enumeration of lattice paths in H.

Lemma 7. *Let $\mathcal{S} \subset \mathbb{Z}^2$ be a finite multiset and let $\mathrm{H}_\theta = \{(x, y) : x \sin\theta + y \cos\theta \geq 0\}$ and let $\mathcal{A}(\theta) = \{i \sin\theta + j \cos\theta : (i, j) \in \mathcal{S}\}$. The combinatorial class $\mathrm{H}_\theta(\mathcal{S})$ is combinatorially isomorphic to $\mathrm{H}(\mathcal{A}(\theta))$.*

Furthermore, if $\mathcal{S}$ is a nontrivial or unrestricted step set for H_θ then the growth constant $K_\mathcal{S}(\theta)$ for the sequence $|\mathrm{H}_\theta(\mathcal{S})_n|$ is the value $K_{\mathcal{A}(\theta)}$ determined by Theorem 2.

Proof. Here it suffices to consider the displacement of each step in the step set in the direction orthogonal to the boundary. See Fig. 2 for an example. The steps $(0, 1)$ and $(1, 0)$ respectively have displacement $\cos\theta$ and $\sin\theta$ in this direction, the other steps follow by linearity. This gives rise to a unidimensional half plane model with step set $\mathcal{A}(\theta)$ to which Theorem 2 applies if $\mathcal{A}(\theta)$ is nontrivial or unrestricted, which occurs precisely when $\mathcal{S}$ is nontrivial or unrestricted in H_θ. $\square$

Example 1 *($\mathcal{S} = \{N, E, SW\} =$ ⅄).* For any $\theta \in [0, \pi/2]$, $\mathrm{Q}(\mathcal{S}) \subseteq \mathrm{H}_\theta(\mathcal{S})$ and the following classes are combinatorially isomorphic

$$\mathrm{H}_\theta(\mathcal{S}) \cong \mathrm{H}(\{\cos\theta, \sin\theta, -\cos\theta - \sin\theta\}).$$

When $\theta \neq \pi/2$, we can scale the model by $\cos\theta^{-1}$. Let $m = \tan\theta, \theta \neq \pi/2$, then $\mathrm{H}_\theta(\mathcal{S}) \cong \mathrm{H}(\{1, m, -m - 1\})$. For $\theta = \pi/2$, remark $\mathrm{H}_{\pi/2}(\mathcal{S}) \cong \mathrm{H}(\{1, 0, -1\})$.

3. Bounds for lattice path models in the quarter plane

As we have already noted in the introduction, the exact enumeration of quarter plane models has been well explored recently. In the case of small steps, Bousquet-Mélou and Mishna identified 79 non-isomorphic, nontrivial small step models [6]. The associated generating functions are known to be D-finite[3] for 23 of these models. After algebraic, the D-finite models are easiest to enumerate asymptotically, and several approaches for this have been successful [4,15,16,3].

The non-D-finite models have been more elusive. Fayolle and Raschel have determined expressions for the growth constant for 74 models [10]. We summarize their formulas in Section 4. Melczer and Mishna determined formulas for singular models [14], and also for highly symmetric models of arbitrary dimension [15].

To describe these formulas we again need the drift of the model, denoted $\delta(\mathcal{S})$:

$$\delta(\mathcal{S}) = \sum_{s\in\mathcal{S}} s = (\delta_x, \delta_y).$$

Here we use a shorthand for classifying drift profiles. For each component, we note if the drift is positive (+), zero (0) or negative (−). For example, if $\delta(\mathcal{S}) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$, the drift profile is $(+, +/0)$.

The *inventory* of the model is the Laurent polynomial $S(x, y)$ defined

$$S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$

This is the two dimensional analog to $A(u)$, and it can be used to express some useful quantities

$$S(1,1) = |\mathcal{S}| \qquad \delta_x = \frac{\partial}{\partial x} S(x,1)\bigg|_{x=1} = S_x(1,1), \qquad \delta_y = \frac{\partial}{\partial y} S(1,y)\bigg|_{y=1} = S_y(1,1).$$

The unidimensional case analysis depended upon the existence of a positive critical point of the inventory. Its existence was a consequence of the nontriviality of the model. The existence in two dimensional case follows from nonsingularity.

We use the result that in the case of a nontrivial, nonsingular model there is a unique solution $(x^*, y^*) \in \mathbb{R}^2_{>0}$ to the equation $S_x(x, y) = S_y(x, y) = 0$. This is a consequence of equation manipulation: If there is no such (x^*, y^*) with x^* and y^* both positive then by a case analysis one can deduce a half plane containing the steps. Alternatively, the reader can refer to Theorem 4 of [5] to see a discussion of this fact in this context. We call this point the *critical point* of the inventory.

[3] The generating functions satisfy linear differential equations with polynomial coefficients.

3.1. Bounds from half plane models

An upper bound on the growth constant of a quarter plane model can always be determined by appealing to a half plane model using the same steps restricted to lie in a region containing the first quadrant. In this section we describe how to determine the half plane which gives the best bound. The main result is Theorem 8. It is followed by examples of its application, its proof, and then in Section 4 a proof that, in the case of small steps, the bound is the same as the exact formula of Fayolle and Raschel. In some cases, this is easy to see, as the upper bound is the same as the lower bound given by excursions.

Theorem 8 *(Main theorem). Let $\mathcal{S} \subset \mathbb{Z}^2$ be a finite multiset that defines a nontrivial quarter plane model $\mathrm{Q}(\mathcal{S})$. Then,*

1. *The growth constant $K_{\mathcal{S}} = \lim_{n\to\infty} q_n^{1/n}$ satisfies*

$$K_{\mathcal{S}} \leq K_{\mathcal{S}}(\theta) \quad \text{for all} \quad 0 \leq \theta \leq \pi/2;$$

where $K_{\mathcal{S}}(\theta)$ is the growth constant for the associated rotated half plane model, as defined in Lemma 7;

2. *The function $K_{\mathcal{S}}(\theta)$ is continuous as a function of θ;*
3. *Let (x^*, y^*) be the positive critical point of the inventory polynomial of the step set $S(x, y)$, if it exists. In the case that $(x^*, y^*) \in [1, \infty)^2$, then $K_{\mathcal{S}}(\theta)$ is minimized at $\theta^* = \arctan \frac{\ln x^*}{\ln y^*}$ with value $K_{\mathcal{S}}(\theta^*) = S(x^*, y^*)$. Otherwise, if (x^*, y^*) is not of this form, or does not exist, then $K_{\mathcal{S}}(\theta)$ is minimized at either 0 or $\pi/2$.*

The minimum being obtained as described does not preclude it also being attained elsewhere. Notably, for many $\mathcal{S}$, $K_{\mathcal{S}}(\theta)$ is a constant function of θ on some interval. More generally, these functions are a smooth convex curve for part of the domain, and then constant on the remainder. Before we prove Theorem 8, we consider some examples to develop some intuition on the behavior.

Example 2 *($\mathcal{S} =$ ).* This is not a singular model. The inventory $S(x, y) = y + \frac{1}{y} + \frac{x}{y} + \frac{1}{xy}$ has a unique critical point at $(1, \sqrt{3})$. The optimal angle given by Theorem 8 is

$$\theta^* = \arctan(\ln(1)/\ln(\sqrt{3})) = 0.$$

Remark $\mathrm{H}(\mathcal{S}) \cong \mathrm{H}(\mathcal{A}(0)) \cong \mathrm{H}(\{1, -1, -1, -1\})$. This leads to the bound $K_{\mathcal{S}} \leq K_{\mathcal{S}}(0) = A_0(\sqrt{3}) = S(1, \sqrt{3}) = 2\sqrt{3}$. This is tight in comparison to the formula given for $K_{\mathcal{S}}$ by Fayolle and Raschel [10]. We shall see that this is a special case of Lemma 11.

Example 3 *($\mathcal{S} =$ ).* This is a nontrivial, singular model. There is no point (x^*, y^*) as in the theorem statement, consequently the maximizing θ is either 0 or $\pi/2$. In fact,

$K_{\mathcal{S}}(\theta)$ is constant in the domain. We deduce $K_{\mathcal{S}} \leq K_{\mathcal{S}}(\theta) = |S| = 4$. This value is tight, according to the formulas of Melczer and Mishna [14]. The other singular cases of Melczer and Mishna behave similarly.

Example 4 *($\mathcal{S} =$ ).* Let us expand this computation to illustrate the process in a generic case. More explicitly, the step set is $\mathcal{S} = \{(0,1), (-1,0), (-1,-1), (0,-1), (1,-1)\}$, with inventory $S(x,y) = y + x/y + 1/y + 1/(xy) + 1/x$. A little bit of calculus reveals that the critical point of P is $(x^*, y^*) \approx (1.6760.., 1.8090..)$. Thus, by the formula, the optimal angle satisfies

$$\theta^* \approx \tan(\ln(1.6760)/\ln(1.8090)) \approx 1.1881..,$$

and the associated rotated half plane model is given by

$$\begin{aligned} A^* &= \{i\sin(\theta^*) + j\cos(\theta^*), (i,j) \in \mathcal{S}\} \\ &= \{0.7540.., -0.6568.., -1.4108.., -0.7540.., -0.0971..\}. \end{aligned}$$

The critical point of the inventory of $\mathcal{A}^*$ is $\tau \approx 2.1950...$ The evaluation of this inventory at τ is 4.2147.. This is precisely the evaluation of S at (x^*, y^*), which is approximately 4.215.

Example 5 *($\mathcal{S} = \{(-2,1), (-2,1), (-1,0), (0,-1), (1,1)\}$).* This model has repeated steps, and a long step. Here we compute the critical point $(\sqrt[3]{7}, \sqrt[3]{7}/3) \approx (1.91, 0.63)$. The minimizing angle here is 0. The best bound is computed $K_{\mathcal{S}}(\theta^*) \approx 3.973$. This approximation agrees with numerical estimates based on the first 200 terms.

Example 4 demonstrates that the best half plane is not defined by the perpendicular to the drift vector (a common hypothesis). Rather, the slope is connected to the Cramér transformation in probability [7]. Denisov and Wachtel assign the probability $p_{ij} = \frac{x^{*i} y^{*j}}{S(x^*,y^*)}$ to the step (i,j) so that the drift of the weighted steps, given by $\sum_{(i,j)\in\mathcal{S}} p_{ij}$, is $(0,0)$, and then apply tools for walks with no drift. It is clear here, perhaps, why their methods apply only to nonsingular walks – they require the existence of x^* and y^*. Bostan, Raschel and Salvy in [5] also discuss the combinatorics of this transform, and show that $S(x^*, y^*)$ is the growth constant for excursions in the quarter plane, which is a subclass of walks, hence $S(x^*, y^*)$ is a lower bound for the growth constant. We offer the interpretation of $S(x^*, y^*)$ as the exponential growth of walks restricted to the half plane H_θ for the angle $\theta = \arctan(\frac{\ln x^*}{\ln y^*})$ (or the right half plane when $y^* = 1$).

3.2. The function $K_{\mathcal{S}}(\theta)$

In the case of half plane walks, the growth constant for the counting sequence for the walks with arbitrary endpoint is either the number of steps, or given by the growth constant for excursions. The deciding factor is the drift. These are the building blocks for functions $K_{\mathcal{S}}(\theta)$.

In our model of changing half planes, the drift is given by the following smooth function of θ:

$$\sum_{(i,j)\in\mathcal{S}} (i\sin\theta + j\cos\theta) = \delta_x \sin\theta + \delta_y \cos\theta.$$

Thus, $K_{\mathcal{S}}(\theta)$ is either the number of steps, or given by $A_\theta(\tau_\theta)$, and switches between them when the drift is 0. Several different possibilities are presented in Fig. 1. Roughly, the function $A_\theta(\tau_\theta)$ is π-periodic and attains a single maximum given by the number of steps, and a single minimum. If that minimum is in the interval $[0, \pi/2]$ it is also the minimum of $K_{\mathcal{S}}(\theta)$.

These functions are well behaved, and we can accurately predict the point where $K_{\mathcal{S}}(\theta)$ attains a minimum. First, we establish the continuity of $K_{\mathcal{S}}(\theta)$ in the next lemma, and then we determine the complete set of critical points, and whether or not they are maxima, or minima, in Lemma 10.

Lemma 9. *Suppose $\mathcal{S} \subset \mathbb{Z}^2$ defines a nontrivial quarter plane model. Then $K_{\mathcal{S}}(\theta)$ defines a continuous function on the domain $\theta \in [0, \pi/2]$.*

Proof. First, we remark that in the domain $\theta \in [0, \pi/2]$, if $\mathcal{S}$ defines a nontrivial quarter plane model, then either $\mathcal{A}_\theta$ defines a nontrivial walk, or an unrestricted walk since it must have at least one step in the quarter plane. The value of $K_{\mathcal{S}}(\theta)$ is defined piecewise according to the value of $\delta(\mathcal{A}(\theta)) = \sum_{(i,j)\in\mathcal{S}}(i\sin\theta + j\cos\theta) = \delta_x \sin\theta + \delta_y \cos\theta$:

$$K_{\mathcal{S}}(\theta) = \begin{cases} |\mathcal{S}| & \text{if } \delta(\mathcal{A}(\theta)) \geq 0 \\ A_\theta(\tau_\theta) & \text{otherwise.} \end{cases}$$

The function $A_\theta(\tau_\theta)$ is continuous as a consequence of Lemma 6 since $A_\theta(\tau_\theta)$ can be expressed as a summation of the form

$$\kappa(\mathbf{x}) = \sum_{j=1}^{\ell} \tau(\mathbf{x})^{x_j},$$

evaluated at an ordering of the elements of $\mathcal{A}(\theta)$. The condition $\delta(\mathcal{A}(\theta)) \geq 0$ defines an interval in $[0, \pi/2]$; consequently $K_{\mathcal{S}}(\theta)$ is piecewise continuous. Finally, as $\delta(\mathcal{A}(\theta))$ approaches 0, $K_{\mathcal{S}}(\theta)$ tends to $|\mathcal{S}|$, by Lemma 5 the function is continuous at points where $\delta(\mathcal{A}(\theta)) = 0$. □

Next, we pinpoint the minimum of $K_{\mathcal{S}}(\theta)$ for θ in the domain $[0, \pi/2]$.

Lemma 10. *Let $\mathcal{S} \subseteq \mathbb{Z}^2$ be the step set of a nontrivial quarter plane model. Let (x^*, y^*) be the positive critical point of the inventory polynomial of the step set $S(x, y)$, if it exists. In the case that $(x^*, y^*) \in [1, \infty)^2$, then $K_\mathcal{S}(\theta)$ is minimized at $\theta^* = \arctan \frac{\ln x^*}{\ln y^*}$ with value $K_\mathcal{S}(\theta^*) = S(x^*, y^*)$. Otherwise, if (x^*, y^*) is not of this form, or does not exist, then $K_\mathcal{S}(\theta)$ is minimized at either 0 or $\pi/2$.*

Proof. Let us rewrite this minimum, using Equation (2.2):

$$\min_{\theta\in[0,\pi/2]} K_\mathcal{S}(\theta) = \min_{\theta\in[0,\pi/2]} \min_{u\geq 1} A_\theta(u). \tag{3.1}$$

Now, $A_\theta(u)$ can be expressed as an evaluation of $S(x, y)$: $A_\theta(u) = S(u^{\sin\theta}, u^{\cos\theta})$. Thus, the latter minimum is equivalent to the following:

$$\min_{\theta\in[0,\pi/2]} \min_{u\geq 1} A_\theta(u) = \min_{x,y\geq 1} S(x, y).$$

Suppose that the minimum of $S(x, y)$ on $[1, \infty)$ is attained at (α, β). We examine the different cases and in each case, we can deduce u^*, θ^* the choice of values for the variables u and θ that minimize $A_\theta(u)$.

If $\alpha = \beta = 1$, then $u^* = 1$ is forced, and any θ gives the same result. If $\alpha > 1$, and $\beta = 1$, then $u^* = 1$ and $\theta^* = 0$, and similarly if $\alpha = 1$, and $\beta > 1$, then $u^* = 1$ and $\theta^* = \pi/2$.

Now, if $\alpha > 1$, and $\beta > 1$ then it is straightforward to deduce that the solution to the system $u^{\sin\theta} = \alpha$, $u^{\cos\theta} = \beta$ has $\theta^* = \arctan(\ln\alpha/\ln\beta)$. Then $K_\mathcal{S}(\theta)$ is minimized at $\theta^* = \arctan(\ln\alpha/\ln\beta)$ with value $S(\alpha, \beta)$.

Suppose $S(x, y)$ has a critical point (x^*, y^*) such that $(x^*, y^*) \in [1, \infty)^2$. Since $S(x, y)$ is convex, it is minimized there. In this case, clearly $(\alpha, \beta) = (x^*, y^*)$. Otherwise, since $S(x, y)$ is a convex function, it will be minimized on the boundary of $[1, \infty)^2$. Thus, either α or β is 1 and so by the cases calculated above the minimum is attained at $\theta = 0$ or $\theta = \pi/2$. This gives the proof. □

Now, we put these ideas together.

Theorem 8. Since $\mathcal{S}$ is a nontrivial model, then A_θ is either a nontrivial or an unrestricted model for all $\theta \in [0, \pi/2]$. For any $\theta \in [0, \pi/2]$, $q_n \leq |\mathrm{H}(\mathcal{A}(\theta))_n|$ for any n, and thus the growth constants of the sequence satisfy $K_\mathcal{S} \leq K_\mathcal{S}(\theta)$, and $K_\mathcal{S}(\theta)$ is minimized at the stated θ^* by Lemma 10. □

4. The case of small steps: a different approach

4.1. The work of Fayolle and Raschel

Fayolle and Raschel [10, Remark 4.9] describe the location of the dominant singularity in the generating function for nonsingular small step quarter plane models. Their formula depends on the drift $\delta(\mathcal{S}) = (\delta_x, \delta_y)$, along with another parameter of the model called the *covariance*. We do not use this parameter, except to compare to their formulas. The covariance of a step set, denoted $\gamma(\mathcal{S})$ is defined as

$$\gamma(\mathcal{S}) = \left.\frac{\partial^2}{\partial x \partial y} S(x, y)\right|_{(x,y)=(1,1)} - \delta_x \delta_y.$$

In the case of small steps the inventory always has the form

$$S(x, y) = a(x)y + b(x) + c(x)y^{-1} = \tilde{a}(y)x + \tilde{b}(y) + \tilde{c}(y)x^{-1}.$$

They prove that there are four possible values for $K_{\mathcal{S}}$:

$$\begin{gathered} |\mathcal{S}|, \qquad \rho_0^{-1} \equiv S(x^*, y^*) \\ \rho_Y^{-1} \equiv b(1) + 2\sqrt{a(1)c(1)}, \qquad \rho_X^{-1} \equiv \tilde{b}(1) + 2\sqrt{\tilde{a}(1)\tilde{c}(1)}. \end{gathered} \tag{4.1}$$

As before, (x^*, y^*) is the unique positive solution in $\mathbb{R}^2_{>0}$ satisfying

$$S_x(x^*, y^*) = 0 \qquad S_y(x^*, y^*) = 0.$$

Furthermore, in their Remark 4.9 they determine conditions on the sign of $\delta(\mathcal{S})$ and $\gamma(\mathcal{S})$ which decide which of the four values is correct for a given nonsingular model.

These results have some natural interpretations. If $\delta(\mathcal{S})$ is non-negative in both components then the growth constant is as for unrestricted walks. If $\delta(\mathcal{S})$ is positive in the first component, and negative in the second, then growth constant is the same as the walks that remain in the upper half plane. The value $S(x^*, y^*)$ is the growth constant for excursions [7,5]. If $\delta(\mathcal{S})$ is negative in both components then the growth constant is the same as the growth constant for excursions in the region. This mirrors the behavior in the case of unidimensional walks. The specific formulas they obtain are simply the result of their cases picking out when $K_{\mathcal{S}}(\theta)$ is minimized at an end point or the critical point. In this way, we are able *unify the six cases*, and deliver *a single interpretation* of the formulas.

Specifically, the next lemma shows how the values $\rho_0, \rho_X, \rho_Y, 1/|\mathcal{S}|$ arise as $K_{\mathcal{S}}(\theta)$, for various θ.

Lemma 11. *For any nontrivial quarter plane model* $\mathrm{Q}(\mathcal{S})$ *with* $\mathcal{S} \subset \{0, \pm 1\}^2$*, the following equalities hold:*

$$\rho_Y^{-1} = A_0(\tau_0) \qquad \rho_X^{-1} = A_{\pi/2}(\tau_{\pi/2}).$$

Proof. This is proved by simply unraveling the notation:

$$A_0(u) = \sum_{(i,j)\in\mathcal{S}} u^j = S(1,u)$$

$$\text{and so}\quad A_0'(u) = [u]S(1,u) - \frac{1}{u^2}[u^{-1}]S(1,u) \Longrightarrow \tau_0 = \sqrt{\frac{[u^{-1}]S(1,u)}{[u]S(1,u)}}.$$

Consequently,

$$A_0(\tau_0) = [y^0]S(1,y) + 2\sqrt{[y]S(1,y)\cdot[y^{-1}]S(1,y)} = \rho_Y^{-1},$$

which is precisely the formula for ρ_Y^{-1} given in Equation (4.1). The ρ_X case is similar since $A_{\pi/2}(u) = \sum_{(i,j)\in\mathcal{S}} u^i$. □

We conclude this section with a discussion that the bounds are tight, i.e.

$$K_{\mathcal{S}} = \min_{\theta\in[0,\pi/2]} K_{\mathcal{S}}(\theta)$$

for all small step quarter plane models. Recall that subsequent to the first version of this document being circulated, this has been proved by Garbit and Raschel, but we can understand it combinatorially.

Theorem 12. *Let $\mathcal{S} \subseteq \{0, \pm 1\}^2$ be a finite set defining a nontrivial, nonsingular quarter-plane lattice path model. The growth constant $K_{\mathcal{S}}$ for the number q_n of walks of length n in $\mathrm{Q}(\mathcal{S})$ satisfies*

$$K_{\mathcal{S}} = \min_{\theta\in[0,\pi/2]} K_{\mathcal{S}}(\theta). \tag{4.2}$$

The location of the minimum is summarized in Table 1.

To prove Theorem 12, we could directly relate the drift profile to the value of $\min_{\theta\in[0,\pi/2]} K_{\mathcal{S}}(\theta)$, and show that this matches the values obtained by Fayolle and Raschel. The different cases to check are summarized in Table 1. One can consider the two equations/inequalities that arise from drift profiles and follow the implications in a straightforward way to deduce the sign of both $\ln x^*$ and $\ln y^*$. Some general case reductions can simplify some of the work. As the inequality manipulations are rather tedious, we do not include them here. It is also possible to simply test the 79 small step cases in order to verify the result. We have done this as well.

Table 1
The value of $K_\mathcal{S}$ for nonsingular nontrivial quarter plane models defined by a finite step set $\mathcal{S} \subset \{\pm 1, 0\}^2$. This value can be computed either using sign information about the drift δ and the covariance γ, or from information about x^* and y^*.

(δ_x, δ_y)	γ	x^*	y^*	$\ln(x^*)/\ln(y^*)$	$\tan(\theta^*)$	$K_\mathcal{S}$
$(+,+)$		< 1	< 1	$+$	$\frac{\ln x^*}{\ln y^*}$	$\lvert\mathcal{S}\rvert$
$(+,0)$						
$(0,+)$						
$(0,0)$		1	1		1	$S(x^*,y^*) = \lvert\mathcal{S}\rvert$
$(0,-)$	$-$	> 1	> 1	$+$	$\frac{\ln x^*}{\ln y^*}$	$S(x^*,y^*)$
	0	1		0	0	$S(x^*,y^*) = \rho_Y^{-1}$
	$+$	< 1		$-$	0	ρ_Y^{-1}
$(+,-)$		< 1		$-$	0	
$(-,0)$	$-$	> 1	> 1	$+$	$\frac{\ln x^*}{\ln y^*}$	$S(x^*,y^*)$
	0		1	∞	∞	$S(x^*,y^*) = \rho_X^{-1}$
	$-$		< 1	$-$	∞	ρ_X^{-1}
$(-,+)$			> 1	$-$	∞	
$(-,-)$		> 1	> 1	$+$	$\frac{\ln x^*}{\ln y^*}$	$S(x^*,y^*)$

5. Extensions and applications

Our half plane bounding strategy does not rely on the size of the steps nor the convex cone in which the paths are restricted. Naive numerical calculations on examples of quarter plane walks with larger steps and of three dimensional models, so far tested in the non-negative octant, suggest the bounds remain tight.

Furthermore, in a recent study of three dimensional walks [2], Bostan, Bousquet-Mélou, Kauers, and Melczer guessed differential equations satisfied by the generating functions of some small step models, and using this they were able to conjecture the exact growth constants. We verified that, in each of the cases for which they had data, the growth constant for the model $O(\mathcal{S})$, where $O = \mathbb{R}^3_{\geq 0}$, is equal to the minimal bound.

These observations led us to the following conjecture.

Conjecture 1. *Let $\mathcal{S} \subset \mathbb{Z}^d$ be a finite multiset of steps. Let $K_\mathcal{S}$ be the growth constant for the enumerative sequence counting the number of walks restricted to the first orthant. Let $\mathcal{P}$ be the set of hyperplanes through the origin in $\mathbb{R}^d$ which do not meet the interior of the first orthant. Given $p \in \mathcal{P}$ let $K_\mathcal{S}(p)$ be the growth constant of the walks on $\mathcal{S}$ which are restricted to the side of p which includes the first orthant. Then*

$$K_\mathcal{S} = \min_{p \in \mathcal{P}} K_\mathcal{S}(p).$$

Note that in all dimensions, $K_\mathcal{S}(p)$ can be computed by projecting the steps onto the normal to p, and enumerating the resulting unidimensional model. Thus, if true, our conjecture would give an elementary way to understand and compute the growth constant of any class of first orthant restricted walks.

The work of Garbit and Raschel translates this into a probabilistic context, and in particular they have proved it in Corollary 9 of [11] for many nontrivial classes. With these same approach, they conjecture [18] that in some cases the sub-exponential growth also matches that of the minimizing half plane, further validating our interpretation, and suggesting a direction for future work.

Our half plane interpretation has important implications for random generation. A naive rejection strategy to generate quarter plane walks might first generate walks in the whole plane, and reject them as they leave the quarter plane. Models with a $(-,-)$ drift profile perform rather poorly in this scheme. However, when the exponential growth rate of a class of quarter plane walks is the same as some class of half plane walks which contain it, the following rejection scheme is provably efficient. The half plane walks with rational slope form an algebraic language, and are easily generated. Rejecting walks from this class when they leave the quarter-plane is remarkably efficient. There are additional details required when the slope is not rational, and the theory is developed by Lumbroso, Mishna and Ponty [13].

Approximations such as we compute here can aid other direct strategies. For example, the strategy of diagonals used by Melczer, Mishna and Wilson relies on determining a set of critical points to set up the integral computations. Having a tight bound on the exponential growth in hand is useful in this process, as some candidates can be eliminated immediately.

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